ACTEX Academic Series

Probability for Risk Management

Third Edition

Matthew J. Hassett, Ph.D

Donald G. Stewart, Ph.D

Jelena Milovanovic, ACIA, AIAA, Ph.D.

3rd

Matthew J. Hassett Ph.D
Donald G. Stewart Ph.D
Jelena Milovanovic ACIA, AIAA, Ph.D

Probability for Risk Management

ACTEX Learning • Greenland, New Hampshire

Request for permission should be addressed to:
ACTEX Learning
PO Box 69
Greenland, NH 03840
support@actexmadriver.com

Manufactured in the United States of America

Cover Design by Rachel Maragnano

Library of Congress Control Number: 2020923382

ISBN: 978-1-64756-323-3

Preface to the Third Edition

Welcome to the third edition of Probability of Risk Management (PRM). The purpose of modernizing this textbook was to bring it up-to-date while retaining the successful style of the past. In this update, we thoroughly revised the text. In particular, the following topics have been enhanced or added: bivariate normal distribution, probability generating functions, coefficient of variation, order statistics, and correlation coefficient. Examples and exercises have also been updated to help better prepare students planning on taking the P-exam.

This text provides a first course in probability for students with a basic calculus background. It has been designed for students who are mostly interested in the applications of the probability to risk management in vital modern areas such as insurance, finance, economics, and health sciences. The text has many features which are tailored for those students and has complete coverage of the SOA Exam P syllabus for those studying for this exam.

The practical and intuitive style of the text is a fundamental feature of this book. Lack of formal proofs does not correspond to a lack of basic understanding. A well-chosen tree example shows most students what Bayes' Theorem is really doing. A simple expected value calculation for the exponential distribution or a polynomial density function demonstrates how expectations are found. We strive to make sure the student feels that he or she understands each concept.

We have also brought on a new author, Jelena Milovanovic, ACIA, AIAA, PHD at Arizona State University. She has brought her teaching knowledge and a fresh perspective to the material to improve the student experience.

The authors would like to thank the third edition review team at ACTEX for their editorial work and continual support on their third round of the text adventure.

Matt Hassett — Tempe, Arizona

Jelena Milovanovic — June 11, 2021

About the Authors

Dr. Matthew J. Hassett is professor emeritus of mathematics at Arizona State University where he taught from 1966 to 1999. He worked in industry as a health actuary from 1999 to 2005, and then returned to Arizona State to continue working with actuarial students. He has published papers in mathematical logic, mathematics education, finance and actuarial science. Dr. Hassett has co-authored an introductory statistics textbook and the ACTEX study guide for Exam M of the Society of Actuaries. He has consulting experience in mathematical analysis of mortgage-backed securities and their derivatives.

Dr. Donald G. Stewart is professor emeritus of mathematics at Arizona State University, following a career of 34 years. His original research interests were in topology. Since 1994, Dr. Steward has worked with actuarial students at ASU. He also participated in the creation of the ASU televised course in Survival Models. Dr. Stewart has extensive experience in providing analysis for expert witnesses. Dr. Stewart has co-authored the ACTEX study manual for Exam M of the Society of Actuaries.

Dr. Jelena Milovanovic is a professor of practice at Arizona State University where she has been teaching since 2003. Originally from Sydney, Australia, she began a career as an actuarial analyst, but discovered early on that her true passion was in academia, thus she obtained a Ph.D. in statistics. Since then, she has been the actuarial science coordinator and one of the founders of the undergraduate and masters' programs at Arizona State. As the faculty advisor for the Gamma Iota Sigma Kappa Chapter at Arizona State, she educates students seeking careers in the insurance industry and helps smooth their transition into the workforce.

Table of Contents

1

Probability: A Tool for Risk Management

1.1 Who Uses Probability?

Probability theory is used for decision-making and risk management throughout modern civilization. Individuals use probability daily, whether or not they know the mathematical theory in this text. If a weather forecaster says that there is a 90% chance of rain, people carry umbrellas. The "90% chance of rain" is a statement of a probability. If a doctor tells a patient that a surgery has a 50% chance of an unpleasant side effect, the patient may want to look at other possible forms of treatment. If a famous stock market analyst states that there is a 90% chance of a severe drop in the stock market, people sell stocks. All of us make decisions about the weather, our finances, and our health based on percentage statements which are really probability statements.

Because probabilities are so important in our analysis of risk, professionals in a wide range of specialties study probability. Weather experts use probability to derive the percentages given in their forecasts. Medical researchers use probability theory in their study of the effectiveness of new drugs and surgeries. Wall Street firms hire mathematicians to apply probability in the study of investments.

The insurance industry has a long tradition of using probability to manage its risks. If you want to buy car insurance, the price you will pay is based on the probability that you will have an accident. This price is called a premium. Life insurance becomes more expensive to purchase as you get older, because there is a higher probability that you will die. Group health insurance rates are based on the study of the probability that the group will have a certain level of claims.

The professionals who are responsible for the risk management and premium calculation in insurance companies are called actuaries. Actuaries take a long series of exams to become certified, and those exams emphasize mathematical probability because of its importance in insurance risk management. Probability is also used extensively in investment analysis, banking, and corporate finance. To illustrate the application of probability in financial risk management, the next section gives a simplified example of how an insurance rate might be set using probabilities.

1.2 An Example from Insurance

In 2009, deaths from motor vehicle accidents occurred at a rate of 10.8 per 100,000 population[1]. This is really a statement of a probability. A mathematician would say that the probability of death from a motor vehicle accident in the next year is $10.8/100{,}000 = .000108$.

Suppose that you decide to sell insurance and offer to pay $10,000 if an insured person dies in a motor vehicle accident. (The money will go to a beneficiary who is named in the policy – perhaps a spouse, a close friend, or the actuarial program at your alma mater.) Your idea is to charge for the insurance and use the money obtained to pay off any claims that may occur. The tricky question is what to charge.

You are optimistic and plan to sell 1,000,000 policies. If you believe the rate of 10.8 deaths from motor vehicles per 100,000 population still holds today, you would expect to have to pay 108 claims on your 1,000,000 policies. You will need 108(10,000) = $1,080,000 to pay those claims. Since you have 1,000,000 policyholders, you can charge each one a premium of $1.08. The charge is small, but 1.08(1,000,000) = $1,080,000 gives you the money you will need to pay claims.

This example is oversimplified. In the real insurance business you would earn interest on the premiums until the claims had to be paid. There are other more serious questions. Should you expect exactly 108 claims from your 1,000,000 clients just because the national rate is 10.8 claims in 100,000? Does the 2009 rate still apply? How can you pay expenses and make a profit in addition to paying claims? To answer these questions requires more knowledge of probability, and that is why this text does not end here. However, the oversimplified example makes a point. Knowledge of probability can be used to pool risks and provide useful goods like insurance. The remainder of this text will be devoted to teaching the basics of probability to students who wish to apply it in areas such as insurance, investments, finance, and medicine.

[1] *Statistical Abstract of the United States*, 2012. Table No. 1103, page 693.

1.3 Probability and Statistics

Statistics is a discipline based on probability that makes inferences from sample data to solve problems. For example, statisticians are responsible for the opinion polls that appear almost every day in the news. In such polls, a sample of a few thousand voters are asked to answer a question such as "Do you think the president is doing a good job?" The results of this sample survey are used to make an inference about the percentage of all voters who think that the president is doing a good job. The insurance problem in Section 1.2 requires use of both probability and statistics. In this text, we will not attempt to teach statistical methods, but we will discuss a great deal of probability theory that is useful in statistics. It is best to defer a detailed discussion of the difference between probability and statistics until the student has studied both areas. It is useful to keep in mind that the disciplines of probability and statistics are related, but not exactly the same.

1.4 Some History

The origins of probability are a piece of everyday life; the subject was developed by people who wished to gamble intelligently. Although games of chance have been played for thousands of years, the development of a systematic mathematics of probability is more recent. Mathematical treatments of probability appear to have begun in Italy in the latter part of the fifteenth century. A gambler's manual, which considered interesting problems in probability, was written by Cardano (1500-1572).

The major advance that led to the modern science of probability was the work of the French mathematician Blaise Pascal. In 1654 Pascal was given a gaming problem by the gambler Chevalier de Mere. The problem of points dealt with the division of proceeds of an interrupted game. Pascal entered into correspondence with another French mathematician, Pierre de Fermat. The problem was solved in this correspondence, and this work is regarded as the starting point for modern probability.

It is important to note that within twenty years of Pascal's work, differential and integral calculus was being developed (independently) by Newton and Leibniz. The subsequent development of probability theory relied heavily on calculus.

Probability theory developed at a steady pace during the eighteenth and nineteenth centuries. Contributions were made by leading scientists such as Jacob Bernoulli, de Moivre, Laplace, Legendre, Gauss and Poisson. Their contributions paved the way for very rapid growth of probability theory in the twentieth century.

Probability is of more recent origin than most of the mathematics covered in university courses. The computational methods of freshman calculus were known in the early 1700's, but many of the probability distributions in this text were not studied until the 1900's. The applications of probability in risk management are even more recent. For example, the foundations of modern portfolio theory were developed by Harry Markowitz [Markowitz, 1952] in 1952. The probabilistic study of mortgage prepayments was developed in the late 1980's to study financial instruments, which were first created in the 1970's and early 1980's.

It would appear that actuaries have a longer tradition of use of probability; a text on life contingencies was published in 1771. However, modern stochastic probability models did not seriously influence the actuarial profession until the 1970's, and actuarial researchers are now actively working with the new methods developed for use in modern finance. The July 2005 copy of the North American Actuarial Journal, which is sitting on my desk, has articles with titles like "Minimizing the Probability of Ruin When Claims Follow Brownian Motion With Drift." You can't read this article unless you know the basics contained in this book and some more advanced topics in probability.

Probability is a young area, with most of its growth in the twentieth century. It is still developing rapidly and being applied in a wide range of practical areas. The history is of interest, but the future will be much more interesting.

1.5 Computing Technology

Modern computing technology has made some practical problems easier to solve. Many probability calculations involve rather difficult integrals; we can now compute these numerically using computers or modern calculators. Some problems are difficult to solve analytically but can be studied using computer simulation. In this text we will give examples of the use of technology in most sections. We will refer to results obtained using the TI-30XS Multiview and TI BA II Plus Professional calculators, Microsoft EXCEL, and R, but will not attempt to teach the use of those tools. The technology sections will be clearly boxed off to separate them from the remainder of the text. Students who do not have the technological background should be aware that this will in no way restrict their understanding of the theory. However, the technology discussions should be valuable to the many students who already use modern calculators or computer packages.

2

Counting for Probability

2.1 What is Probability?

People who have never studied the subject understand the intuitive ideas behind the mathematical concept of probability. Teachers (including the authors of this text) usually begin a probability course by asking the students if they know the probability of a coin toss coming up heads. The obvious answer is 50% or 1/2, and most people give the obvious answer with very little hesitation. The reasoning behind this answer is simple. There are two possible outcomes of the coin toss, heads or tails. If the coin comes up heads, only one of the two possible outcomes has occurred. There is one chance in two of tossing a head.

The simple reasoning here is based on an assumption – the coin must be fair, so that heads and tails are **equally likely**. If your gambler friend Fast Eddie invites you into a coin tossing game, you might suspect that he has altered the coin so that he can get your money. However, if you are willing to assume that the coin is fair, you count possibilities and come up with 1/2.

Probabilities are evaluated by counting in a wide variety of situations. Gambling related problems involving dice and cards are typically solved using counting. For example, suppose you are rolling a fair single six-sided die. You wish to bet on the event that you will roll a number less than 5. The probability of this event is 2/3 since the outcomes 1, 2, 3 and 4 are less than 5, and there are six possible outcomes (assumed equally likely). The approach to probability used is summarized as follows:

Probability by Counting for Equally Likely Outcomes

$$\text{Probability of an event} = \frac{\text{Number of outcomes in the event}}{\text{Total number of possible outcomes}}$$

Part of the work of this chapter will be to introduce a more precise mathematical framework for this counting definition. However, this is not the only way to view probability. There are some cases in which outcomes may not be equally likely. A die or a coin may be altered so that all outcomes are not equally likely. Suppose that you are tossing a coin and suspect that it is not fair. Then the probability of tossing a head cannot be determined by counting, but there is a simple way to estimate that probability – simply toss the coin a large number of times and count the number of heads. If you toss the coin 1000 times and observe 650 heads, your best estimate of the probability of a head on one toss is $650/1{,}000 = .65$. In this case you are using a **relative frequency estimate** of a probability.

Relative Frequency Estimate of the Probability of an Event

$$\text{Probability of an event} = \frac{\text{Number of favorable outcomes}}{\text{Total number trials}}$$

We now have two ways of looking at probability: the counting approach for equally likely outcomes and the relative frequency approach. This raises an interesting question. If outcomes are equally likely, will both approaches lead to the same probability? For example, if you try to find the probability of tossing a head for a fair coin by tossing the coin a large number of times, should you expect to get a value of 1/2? The answer to this question is "not exactly, but for a very large number of tosses you are highly likely to get an answer close to 1/2." The more tosses, the more likely you are to be very close to 1/2. We used R to simulate different numbers of coin tosses, and came up with the following results.

Number of Tosses	Number of Heads	Probability Estimate
4	1	.2500
100	54	.5400
1000	524	.5240
10000	4985	.4985

More will be said later in the text about the mathematical reasoning underlying the relative frequency approach. Many texts identify a third approach to probability. That is the **subjective** approach to probability. Using this approach, you ask a well-informed person for his or her personal estimate of the probability of an event. For example, one of your authors worked on a business valuation problem which required knowledge of the probability that an individual would fail to make a monthly mortgage payment to a company. He went to an executive of the company and asked what percent of individuals failed to make the monthly payment in a typical month. The executive, relying on his experience, gave an estimate of 3%, and the valuation problem was solved using a subjective probability of .03. The executive's subjective estimate of 3% was based on a personal recollection of relative frequencies he had seen in the past.

In the remainder of this chapter we will work on building a more precise mathematical framework for probability. The counting approach will play a big part in this framework, but the reader should keep in mind that many of the probability numbers actually used in calculation may come from relative frequencies or subjective estimates.

2.2 The Language of Probability: Sets, Sample Spaces, and Events

If probabilities are to be evaluated by counting outcomes of a probability experiment, it is essential that all outcomes be specified. A person who is not familiar with dice does not know that the possible outcomes for a single fair die are 1, 2, 3, 4, 5 and 6. That person cannot find the probability of rolling a 1 with a single die because the basic outcomes are unknown. In every well-defined probability experiment, all possible outcomes must be specified in some way.

The language of set theory is very useful in the analysis of outcomes. Sets are covered in most modern mathematics courses, and the reader is assumed to be familiar with some set theory. For the sake of completeness, we will review some of the basic ideas of set theory. A **set** is a collection of objects such as the numbers 1, 2, 3, 4, 5 and 6. These objects are called the **elements** or **members** of the set. If the set is finite and small enough that we can easily list all of its elements, we can describe the set by listing all of its elements in braces. For the set above, $S = \{1, 2, 3, 4, 5, 6\}$. For large or infinite sets, the set-builder notation is helpful. For example, the set of all positive real numbers may be written as

$$S = \{x \mid x \text{ is a real number and } x > 0\}.$$

Often it is assumed that the numbers in question are real numbers, and the set above is written as $S = \{x \mid x > 0\}$.

We will review more set theory as needed in this chapter. The important use of set theory here is to provide a precise language for dealing with the outcomes in a probability experiment. The definition below uses the set concept to refer to all possible outcomes of a probability experiment.

Definition 2.1. The **sample space** for a probability experiment is the set of all possible outcomes of the experiment, and is usually denoted by S.

Example 2.1. A single die is rolled and the number facing up recorded. The sample space is $S = \{1, 2, 3, 4, 5, 6\}$. □

Example 2.2. A coin is tossed and the side facing up is recorded. The sample space is $S = \{H, T\}$. □

Many interesting applications involve a simple two-element sample space. The following examples are of this type.

Example 2.3. An insurance company is interested in the probability that an insured will die in the next year. The sample space is $S =$\{*death*, *survival*\}. □

Example 2.4. A manufacturer is interested in the probability that a crucial part in a machine will fail in the next week. The sample space is $S =$\{*failure*, *survival*\}. □

Example 2.5. Companies borrow money they need by issuing bonds. A bond is typically sold in \$1000 units that have a fixed interest rate such as 8% per year for twenty years. When you buy a bond for \$1000, you are actually loaning the company your \$1000 in return for 8% interest per year. You are supposed to get your \$1000 loan back in twenty years. If the company issuing the bonds has financial trouble, it may declare bankruptcy and default by failing to pay your money back. Investors who buy bonds wish to find the probability of default. The sample space is $S =$ \{*default*, *no default*\}. □

Example 2.6. Homeowners usually buy their homes by getting a mortgage loan that is repaid by monthly payments. The homeowner usually has the right to pay off the mortgage loan early if that is desirable – because the homeowner decides to move and sell the house, because interest rates have gone down, or because someone has won the lottery. Lenders may lose or gain money when a loan is prepaid early, so they are interested in the probability of prepayment. If the lender is interested in whether the loan will prepay in the next month, the sample space is $S =$\{*prepayment*, *no prepayment*\}. □

The simple sample spaces above are all of the same type. Something (a bond, a mortgage, a person, or a part) either continues or disappears. Despite this deceptive simplicity, the probabilities involved are of great importance. If a part in your airplane fails, you may become an insurance death – leading to the prepayment of your mortgage using your death benefit and thus putting a strain on your insurance company and its bonds.

The probabilities are difficult and costly to estimate. Note also that the coin toss sample space $\{H, T\}$ was the only one in which the two outcomes were equally likely. Luckily for most of us, insured individuals are more likely to live than die, and bonds are more likely to succeed than to default.

Not all sample spaces are so small or so simple.

Example 2.7. An insurance company has sold 100 individual life insurance policies. When an insured individual dies, the beneficiary named in the policy will file a claim for the amount of the policy. You wish to observe the number of claims filed in the next year. The sample space consists of all integers from 0 to 100, so $S = \{0, 1, 2, \ldots, 100\}$. □

Some of the previous examples may be looked at in slightly different ways that lead to different sample spaces. The sample space is determined by the question you are asking.

Example 2.8. An insurance company sells life insurance to a 30-year-old female. The company is interested in the age of the insured when she eventually dies. If the company assumes that the insured will not live to 110, the sample space is $S = \{30, 31, \ldots, 109\}$. □

Example 2.9. A mortgage lender makes a 30-year monthly payment loan. The lender is interested in studying the month in which the mortgage is paid off. Since there are 360 months in 30 years, the sample space is $S = \{1, 2, 3, \ldots, 359,\ 360\}$. □

The sample space can also be infinite.

Example 2.10. A stock is purchased for \$100. You wish to observe the price it can be sold for in one year. Since stock prices are quoted in dollars and fractions of dollars, the stock could have any non-negative rational number as its future value. The sample space consists of all non-negative rational numbers, $S = \{x \mid x \geq 0 \text{ and } x \text{ rational}\}$. This does not imply that the price outcome of \$1,000,000,000 is highly likely in one year – just that it is possible. Note that the price outcome of 0 is also possible. Stocks can become worthless. □

The above examples show that the sample space for an experiment can be a small finite set, a large finite set, or an infinite set.

In Section 2.1 we looked at the probability of events which were specified in words, such as "toss a head" or "roll a number less than 5". These events also need to be translated into clearly specified sets. For example, if a single die is rolled, the event "roll a number less than 5" consists of the outcomes in the set $E = \{1, 2, 3, 4\}$. Note that the set E is a subset of the sample space S, since every element of E is an element of S. This leads to the following set-theoretical definition of an event.

Definition 2.2. Any subset E of the sample space S is called an event.

This set-theoretic definition of an event often causes some unnecessary confusion since people think of an event as something described in words like "roll a number less than 5 on a roll of a single die". There is no conflict here. The definition above reminds you that you must take the event described in words and determine precisely what outcomes are in the event. Below we give a few examples of events that are stated in words and then translated into subsets of the sample space.

Example 2.11. A coin is tossed. You wish to find the probability of the event "toss a head". The sample space is $S = \{H, T\}$. The event is the subset $E = \{H\}$. □

Example 2.12. An insurance company has sold 100 individual life policies. The company is interested in the probability that at most 5 of the policies have death benefit claims in the next year. The sample space is $S = \{0, 1, 2, \ldots, 100\}$. The event E is the subset $\{0, 1, 2, 3, 4, 5\}$. □

Example 2.13. You buy a stock for \$100 and plan to sell it one year later. You are interested in the event that you make a profit when the stock is sold. The sample space is $S = \{x \mid x \geq 0 \text{ and } x \text{ rational}\}$, the set of all possible future prices. The event is the subset $E = \{x \mid x > 100 \text{ and } x \text{ rational}\}$, the set of all possible future prices that are greater than the \$100 you paid. □

Problems involving selections from a standard 52 card deck are common in beginning probability courses. Such problems reflect the origins of probability. To make listing simpler in card problems, we will adopt the following abbreviation system:

A – Ace	K – King	Q – Queen	J – Jack
S – Spade	H – Heart	D – Diamond	C – Club

We can then describe individual cards by combining letters and numbers. For example KH will stand for the king of hearts and $2D$ for the 2 of diamonds.

Example 2.14. A standard 52 card deck is shuffled, and a card is picked at random. You are interested in the event that the card is a king. The sample space, $S = \{AS, KS, \ldots, 3C, 2C\}$, consists of all 52 cards. The event E consists of the four kings, $E = \{KS, KH, KD, KC\}$. □

The examples of sample spaces and events given above are straight forward. In many practical problems things become much more complex. The following sections introduce more set theory and some counting techniques that will help in analyzing more difficult problems.

2.3 Compound Events; Set Notation

When we refer to events in ordinary language, we often negate them (the card drawn is *not* a king) or combine them using the words "and" or "or" (the card drawn is a king *or* an ace). Set theory has a convenient notation for use with such **compound events**.

2.3.1 Negation

The event *not* E is written as $\sim E$. (This may also be written as $\overline{E}$.)

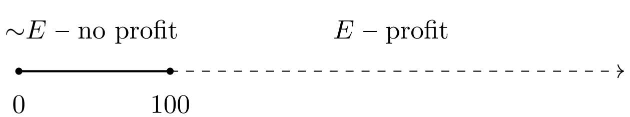
Example 2.15. A single die is rolled, $S = \{1, 2, 3, 4, 5, 6\}$. The event E is the event of rolling a number less than 5, so $E = \{1, 2, 3, 4\}$. E does not occur when a 5 or 6 is rolled. Thus $\sim E = \{5, 6\}$. □

Note that the event $\sim E$ is called the **complement** of E and is the set of all outcomes in the sample space that are not in the original event set E.

Example 2.16. You buy a stock for \$100 and wish to evaluate the probability of selling it for a higher price x in one year.
The sample space is $S = \{x \mid x \geq 0 \text{ and } x \text{ rational}\}$.
The event of interest is $E = \{x \mid x > 100 \text{ and } x \text{ rational}\}$.
The negation $\sim E$ is the event that no profit is made on the sale,
so $\sim E$ can be written as

$$\sim E = \{x \mid 0 \leq x \leq 100 \text{ and } x \text{ rational}\}.$$

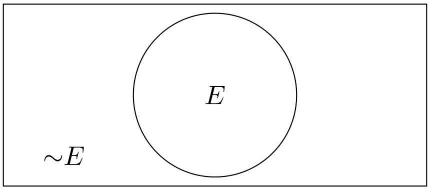
This can be portrayed graphically on a number line.



□

Graphical depiction of events is very helpful. The most common tool for this is the **Venn diagram** in which the sample space is portrayed as a rectangular region, and the event is portrayed as a circular region inside the rectangle.

The Venn diagram showing E and $\sim E$ is given in the following figure.



2.3.2 The Compound Events E or F, E and F

We will begin by returning to the familiar example of rolling a single die. Suppose that we have the opportunity to bet on two different events:

E – an even number is rolled $\quad$ F – a number less than 5 is rolled

$E = \{2, 4, 6\}$ $\quad$ $F = \{1, 2, 3, 4\}$

If we bet that E or F occurs, we will win if any element of the two sets above is rolled.

$$E \text{ or } F = \{1, 2, 3, 4, 6\}$$

In forming the set for E or F we have combined the sets E and F by listing all outcomes that appear in *either* E or F. The resulting set is called the **union** of E and F and is written as $E \cup F$. The union for any two events E and F can be represented as

$$E \text{ or } F = E \cup F.$$

For the single die roll above, we could also decide to bet on the event E and F. In that case, *both* the event E and the event F must occur on the single roll. This can happen only if an outcome occurs that is common to both events.

$$E \text{ and } F = \{2, 4\}$$

In forming the set E and F we have listed all outcomes that are in both sets simultaneously. This set is referred to as the **intersection** of E and F and is written as $E \cap F$. The intersection for any two events E and F

can be represented as

$$E \text{ and } F = E \cap F.$$

Example 2.17. Consider the insurance company that has written 100 individual life insurance policies and is interested in the number of claims that will occur in the next year. The sample space is $S = \{0, 1, 2, \ldots, 100\}$. The company is interested in the following two events:

E – there are at most 8 claims

F – the number of claims is between 5 and 12 (inclusive)

E and F are given by the sets

$$E = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$$

and

$$F = \{5, 6, 7, 8, 9, 10, 11, 12\}.$$

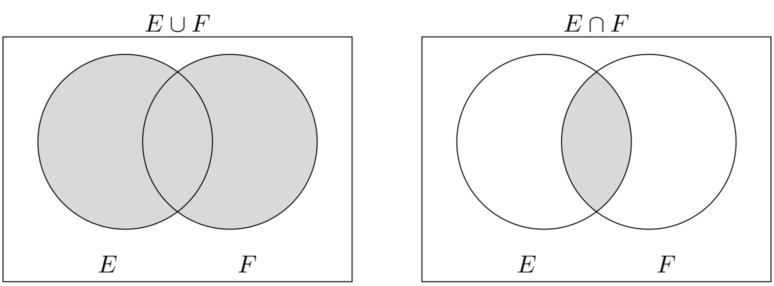
Then the events (E or F) and (E and F) are given by

$$E \text{ or } F = E \cup F = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$$

and

$$E \text{ and } F = E \cap F = \{5, 6, 7, 8\}. \quad \square$$

The events (E or F) and (E and F) can also be represented using Venn diagrams, with overlapping circular regions representing E and F.



2.3.3 New Sample Spaces from Old; Ordered Pair Outcomes

In some situations the basic outcomes of interest are actually pairs of simpler outcomes. The following examples illustrate this.

Example 2.18. Life insurance is often written on a couple. Suppose the insurer is interested in whether one or both members of the couple die in the next year. Then the insurance company must start by considering the following outcomes:

D_1 – death of Partner 1 S_1 – survival of Partner 1

D_2 – death of Partner 2 S_2 – survival of Partner 2

Since the insurance company has written a policy insuring both partners, the sample space of interest consists of pairs that show the status of both Partner 1 and Partner 2. For example, the pair (D_1, S_2) describes the outcome in which Partner 1 dies but Partner 2 survives. The sample space is

$$S = \{(D_1, S_2), (D_1, D_2), (S_1, S_2), (S_1, D_2)\} .$$

In this sample space, events may be more complicated than they sound. Consider the following event:

$$P1 \text{ – Partner 1 dies in the next year,}$$
$$P1 = \{(D_1, S_2\}, (D_1, D_2)\} .$$

The death of Partner 1 is not a single outcome. The insurance company has insured two people and has different obligations for each of the two outcomes in $P1$. The death of Partner 2 is similar.

$$P2 \text{ – Partner 2 dies in the next year,}$$
$$P2 = \{(D_1, D_2), (S_1, D_2)\} .$$

The events $P1$ or $P2$ and $P1$ and $P2$ are also sets of pairs.

$$P1 \cup P2 = \{(D_1, S_2), (D_1, D_2), (S_1, D_2)\} ,$$
$$P1 \cap P2 = \{(D_1, D_2)\} .$$ □

Similar reasoning can be used in the study of the failure of two crucial parts in a machine or the prepayment of two mortgages.

2.4 Set Identities

2.4.1 The Distributive Laws for Sets

The distributive law for real numbers is the familiar

$$a(b + c) = ab + ac.$$

Two similar distributive laws for set operations are the following:

$$E \cap (F \cup G) = (E \cap F) \cup (E \cap G) \tag{2.1}$$

$$E \cup (F \cap G) = (E \cup F) \cap (E \cup G) \tag{2.2}$$

These laws are helpful in dealing with compound events involving the connectives *and* and *or*. They tell us that

E and (F or G) is equivalent to (E and F) or (E and G)

and

E or (F and G) is equivalent to (E or F) and (E or G).

The validity of these laws can be seen using Venn diagrams. This is pursued in the exercises. These identities are illustrated in the following example.

Example 2.19. A financial services company is studying a large pool of individuals who are potential clients. The company offers to sell its clients stocks, bonds, and life insurance. The events of interest are the following:

S – the individual owns stocks

B – the individual owns bonds

I – the individual has life insurance coverage

The distributive laws tell us that

$$I \cap (B \cup S) = (I \cap B) \cup (I \cap S)$$

and

$$I \cup (B \cap S) = (I \cup B) \cap (I \cup S).$$

The first identity states that

insured *and* (owning bonds *or* stocks)
is equivalent to
(insured *and* owning bonds) *or* (insured *and* owning stocks).

The second identity states that

insured *or* (owning bonds *and* stocks)
is equivalent to
(insured *or* owning bonds) *and* (insured *or* owning stocks).

2.4.2 De Morgan's Laws

Two other useful set identities are the following:

$$\sim(E \cup F) = (\sim E \cap \sim F) \tag{2.3}$$

$$\sim(E \cap F) = (\sim E \cup \sim F) \tag{2.4}$$

These laws state that

not(E or F) is equivalent to (not E) and (not F)

and

not(E and F) is equivalent to (not E) or (not F).

As before, verification using Venn diagrams is left for the exercises. The identity is seen more clearly through an example.

Example 2.20. We return to the events S (ownership of stock) and B (ownership of bonds) in the previous example. De Morgan's laws state that

$$\sim(S \cup B) = \sim S \cap \sim B$$

and

$$\sim(S \cap B) = \sim S \cup \sim B.$$

In words, the first identity states that if you don't own stocks *or* bonds, then you don't own stocks *and* you don't own bonds (and vice versa). The second identity states that if you don't own both stocks *and* bonds, then you don't own stocks *or* you don't own bonds (and vice versa). □

De Morgan's laws and the distributive laws are worth remembering. They enable us to simplify events that are stated verbally or in set notation. They will be useful in the counting and probability problems that follow.

2.5 Counting

Since many (not all) probability problems will be solved by counting outcomes, this section will develop a number of counting principles that will prove useful in solving probability problems.

2.5.1 Basic Rules

We will first illustrate the basic counting rules by example and then state the general rules. In counting, we will use the convenient notation

$$n(E) = \text{ the number of elements in the set (or event) } E.$$

Example 2.21. A neighborhood association has 100 families on its membership list. 70 of the families have a credit card,[2] and 50 of the families are currently paying off a car loan. 41 of the families have both a credit card and a car loan. A financial planner intends to call on one of the 100 families today. The planner's sample space consists of the 100 families in the association. The events of interest to the planner are the following:

C – the family has a credit card L – the family has a car loan

We are given the following information:

$$n(C) = 70 \qquad n(L) = 50 \qquad n(L \cap C) = 41$$

The planner is also interested in the answers to some other questions. For example, the planner would first like to know how many families do not have credit cards. Since there are 100 families and 70 have credit cards, the number of families that do not have credit cards is $100 - 70 = 30$.

[2] In 2007, 70.2% of American families had credit cards. ([U.S. Census Bureau, 2012], Table No. 1189.)

This can be written using our counting notation as

$$n(\sim C) = n(S) - n(C). \qquad \square$$

This reasoning clearly works in all situations, giving the following general rule for any finite sample space S and event E.

$$n(\sim E) = n(S) - n(E) \tag{2.5}$$

Example 2.22. The planner in the previous example would also like to know how many of the 100 families had a credit card or a car loan. If the planner adds $n(C) = 70$ and $n(L) = 50$, the result of 120 is clearly too high. This happened because in the 120 figure each of the 41 families with both a credit card and a car loan was counted twice. To reverse the double counting and get the correct answer, subtract 41 from 120 to get the correct count of 79. This is written below in our counting notation.

$$n(C \cup L) = n(C) + n(L) - n(C \cap L) = 70 + 50 - 41 = 79. \qquad \square$$

The reasoning in Example 2.22 also applies in general to any two events and in any finite sample space and is referred to as the **general addition rule**.

$$n(E \cup F) = n(E) + n(F) - n(E \cap F) \tag{2.6}$$

Example 2.23. A single card is drawn at random from a well shuffled deck. The events of interest are the following:

H – the card drawn is a heart	$n(H) = 13$
K – the card is a king	$n(K) = 4$
C – the card is a club	$n(C) = 13$

The compound event $H \cap K$ occurs when the card drawn is both a heart and a king (i.e., the card is the king of hearts). Then $n(H \cap K) = 1$ and

$$n(H \cup K) = n(H) + n(K) - n(H \cap K) = 13 + 4 - 1 = 16.$$

The situation is somewhat simpler if we look at the events H and C. Since a single card is drawn, the event $H \cap C$ can only occur if the single card drawn is both a heart and a club, which is impossible. There are no outcomes in $H \cap C$, and $n(H \cap C) = 0$. Then

$$n(H \cup C) = n(H) + n(C) - n(H \cap C) = 13 + 13 - 0 = 26.$$

More simply,

$$n(H \cup C) = n(H) + n(C).$$

□

The two events H and C are called **mutually exclusive** because they cannot occur together. The occurrence of H excludes the possibility of C and vice versa. There is a convenient way to write this in set notation.

Definition 2.3. The **empty set** is the set that has no elements and is denoted by the symbol $\emptyset$.

In the above example, we could write $H \cap C = \emptyset$ to show that H and C are mutually exclusive. The same principle applies in general.

Definition 2.4. Two events E and F are mutually exclusive if $E \cap F = \emptyset$.

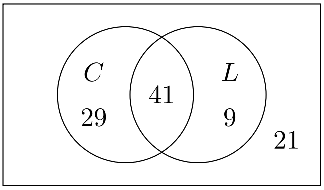
If E and F are mutually exclusive, then

$$n(E \cup F) = n(E) + n(F) \tag{2.7}$$

2.5.2 Using Venn Diagrams in Counting Problems

Venn diagrams are helpful in visualizing all of the components of a counting problem. This is illustrated in the following example.

Example 2.24. The following Venn diagram is labeled to completely describe all of the components of Example 2.21. In that example the sample space consisted of 100 families. Recall that the events of interest were C (the family has a credit card) and L (the family has a car loan). We were given that $n(C) = 70$, $n(L) = 50$ and $n(L \cap C) = 41$. We found that $n(L \cup C) = 79$. The Venn diagram below shows all this and more.



Since $n(C) = 70$ and $n(L \cap C) = 41$, there are 70 families with credit cards and 41 families with both a credit card and a car loan. This leaves $70 - 41 = 29$ families with a credit card and no car loan. We write the number 29 in the part of the region for C that does not intersect L. Since $n(L) = 50$, there are only 9 families with a car loan and no credit card, so we write 9 in the appropriate region. The total number of families with either a credit card or a car loan is clearly given by $29 + 41 + 9 = 79$. Finally, since $n(S) = 100$, there are $100 - 79 = 21$ families with neither a credit card nor a car loan. □

The numbers on the previous page could all be derived using set identities and written in the following set theoretic terms:

$$n(L \cap C) = 41,$$

$$n(\sim L \cap C) = 29,$$

$$n(L \cap \sim C) = 9,$$

$$n(\sim L \cap \sim C) = 21.$$

The Venn diagram can also be used in counting problems involving three events but requires a more complicated diagram as the following example shows.

Example 2.25. A survey of 120 students was conducted at a small college, and it was discovered that 60 students took English, 56 took Mathematics, and 42 took Chemistry. 82 students took English or Mathematics, 86 took English or Chemistry, and 78 took Mathematics or Chemistry. 6 students took all three courses. How many students took exactly two courses?

Solution. The events of interest are the following:

E – student took English	$n(E) = 60$
M – student took Mathematics	$n(M) = 56$
C – student took Chemistry	$n(C) = 42$

The compound events given in the question are

$$n(E \cup M) = 82,\ n(E \cup C) = 86,\ n(C \cup M) = 78, \text{ and } n(E \cap C \cap M) = 6.$$

Using the general addition rule to find $n(E \cap M)$, we have

$$n(E \cup M) = n(E) + n(M) - n(E \cap M),$$

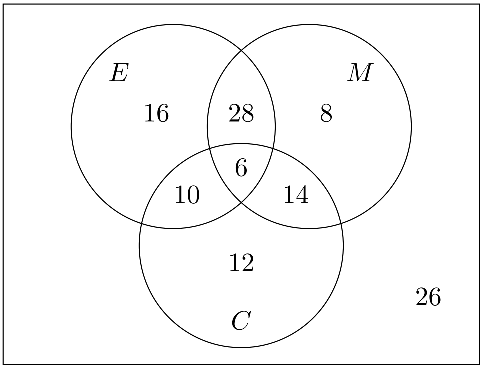
so

$$82 = 60 + 56 - n(E \cap M)$$

and therefore

$$n(E \cap M) = 34.$$

Applying the same strategy, we find $n(E \cap C) = 16$ and $n(M \cap C) = 20$. Out of the 34 students who took English and Mathematics, 6 students also took Chemistry, leaving 28 students who took *only* English and Mathematics. Similarly, 10 students took *only* English and Chemistry, and 14 students took *only* Mathematics and Chemistry. Therefore, $28 + 10 + 14 = 52$ students took exactly two courses at the small college. The Venn diagram below shows all this and more. □



It is worth noting that the general addition rule can be extended to more than two events. Since three-event problems are frequently encountered, below is the corresponding general addition rule.

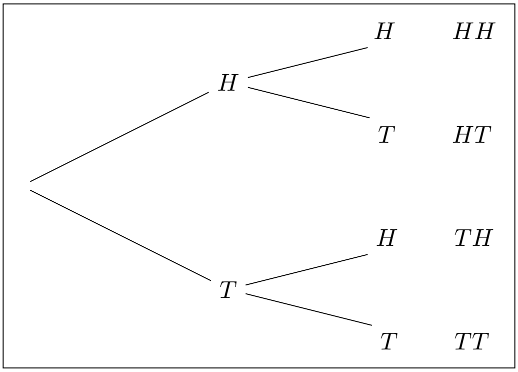
$$\begin{aligned} n(E \cup F \cup G) = &n(E) + n(F) + n(G) \\ &- n(E \cap F) - n(E \cap G) - n(F \cap G) \\ &+ n(E \cap F \cap G) \end{aligned} \tag{2.8}$$

It is worth a few seconds to check this identity in the preceding example.

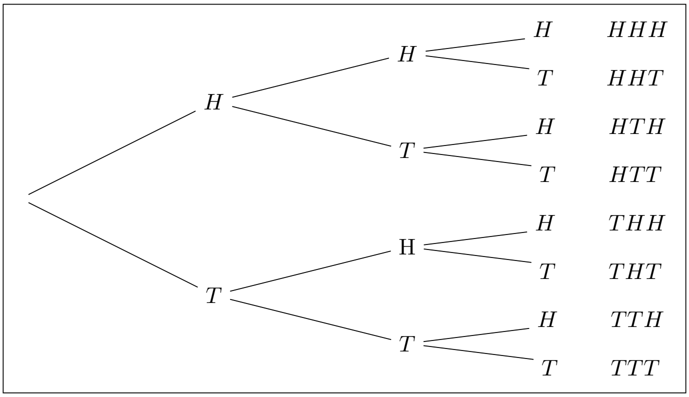
2.5.3 Trees

A **tree** gives a graphical display of all possible cases in a problem.

Example 2.26. A coin is tossed twice. The tree that gives all possible outcomes is shown below. We create one branch for each of the two outcomes on the first toss, and then we attach a second set of branches to each of the first to show the outcomes on the second toss. The results of the two tosses along each set of branches are listed at the right of the diagram. □



A tree provides a simple display of all possible pairs of outcomes in an experiment *if the number of outcomes is not unreasonably large.* It would not be reasonable to attempt a tree for an experiment in which two numbers between 1 and 100 were picked at random, but it is reasonable to use a tree to show the outcomes for three successive coin tosses. Such a tree is shown next.



Trees will be used extensively in this text as visual aids in problem solving. Many problems in risk analysis can be better understood when all possibilities are displayed in this fashion. The next example gives a tree for disease testing.

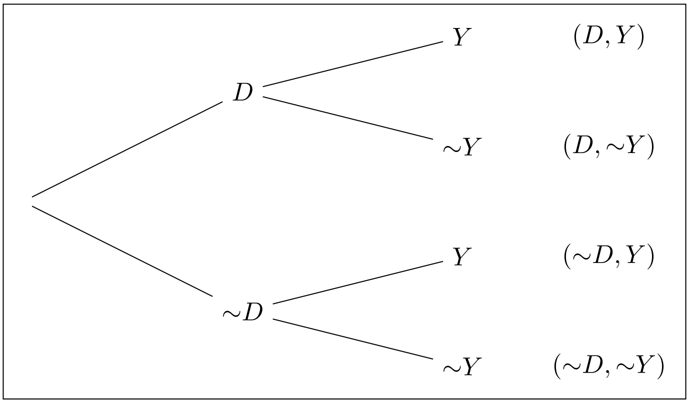
Example 2.27. A test for the presence of a disease has two possible outcomes – positive or negative. A positive outcome indicates that the tested person *may* have the disease, and a negative outcome indicates that the tested person probably does not have the disease. Note that the test is not perfect. There may be some misleading results. The possibilities are shown in the tree below. We have the following outcomes of interest:

D – the person tested has the disease

$\sim D$ – the person tested does not have the disease

Y – the test is positive

$\sim Y$ – the test is negative



The outcome $(\sim D, Y)$ is referred to as a **false positive** result. The person tested does not have the disease, but nonetheless tests positive for it. The outcome $(D, \sim Y)$ is a **false negative** result. □

2.5.4 The Multiplication Principle for Counting

The trees in the prior section illustrate a fundamental counting principle. In the case of two coin tosses, there were two choices for the outcome at the end of the first branch, and for each outcome on the first toss there were two more possibilities for the second branch. This led to a total of $2 \times 2 = 4$ outcomes. This reasoning is a particular instance of a very useful general law.

The Multiplication Principle for Counting

Suppose that the outcomes of an experiment consist of a combination of two separate tasks or actions. Suppose there are n possibilities for the first task, and that for each of these n possibilities there are k possible ways to perform the second task. Then there are nk possible outcomes for the experiment.

Example 2.28. A coin is tossed twice. The first toss has $n = 2$ possible outcomes, and the second toss has $k = 2$ possible outcomes. The experiment (two tosses) has $nk = 2 \times 2 = 4$ possible outcomes. □

Example 2.29. An employee of a southwestern state can choose one of three group life insurance plans and one of five group health insurance plans. The total number of ways she can choose her complete life and health insurance package is $3 \times 5 = 15$. □

The validity of this counting principle can be seen by considering a tree for the combination of tasks. There are n possibilities for the first branch, and for each first branch there are k possibilities for the second branch. This will lead to a total of nk combined branches. Another way to present the rule schematically is the following:

Task 1	Task 2	Total Outcomes
n ways	k ways	nk ways

The multiplication principle also applies to combined experiments consisting of more than two tasks. Following Example 2.26, we gave a tree to show all possible outcomes of tossing a coin three times. There were $2 \times 2 \times 2 = 8$ total outcomes for the combined experiment. This illustrates the general multiplication principle for counting.

Suppose that the outcomes of an experiment consist of a combination of k separate tasks or actions. If task i can be performed in n_i ways for each combined outcome of the remaining tasks for $i = 1, \ldots, k$, then the total number of outcomes for the experiment is $n_1 \times n_2 \times \cdots \times n_k$. Schematically, we have the following:

Task 1	Task 2	$\cdots$	Task k	Total Outcomes
n_1	n_2	$\cdots$	n_k	$n_1 \times n_2 \times \cdots \times n_k$

Example 2.30. A certain mathematician owns 8 pairs of socks, 4 pairs of pants, and 10 shirts. The number of different ways he can get dressed is $8 \times 4 \times 10 = 320$. (It is important to note that this solution only

applies if the mathematician will wear anything with anything else, which is a matter of concern to his wife.) □

The number of total possibilities in an everyday setting can be surprisingly large.

Example 2.31. A restaurant has 9 appetizers, 12 main courses, and 6 desserts. Each main course comes with a salad, and there are 6 choices for salad dressing. The number of different meals consisting of an appetizer, a salad with dressing, a main course, and a dessert is therefore $9 \times 6 \times 12 \times 6 = 3,888$. □

2.5.5 Permutations

In many practical situations it is necessary to arrange objects in order. If you were considering buying one of four different cars, you would be interested in a 1, 2, 3, 4 ranking that ordered them from best to worst. If you are scheduling a meeting in which there are 5 different speakers, you must create a program that gives the order in which they speak.

Definition 2.5. A **permutation** of n objects is an ordered arrangement of those objects.

The number of permutations of n objects can be found using the multiplication principle.

Example 2.32. The number of ways that four different cars can be ranked is shown schematically below.

Rank 1	Rank 2	Rank 3	Rank 4	Total Ways to Rank
4	3	2	1	$4 \times 3 \times 2 \times 1 = 24$

The successive tasks here are to choose Ranks 1, 2, 3 and 4. At the beginning there are 4 choices for Rank 1. After the first car is chosen, there are 3 cars left for Rank 2. After 2 cars have been chosen, there are only 2 cars left for Rank 3. Finally, there is only one car left for Rank 4. □

The same reasoning works for the problem of arranging 5 speakers in order. The total number of possibilities is $5 \times 4 \times 3 \times 2 \times 1 = 120$. To handle problems like this, it is convenient to use factorial notation.

$$n! = n(n-1)(n-2)\cdots(2)(1).$$

The notation $n!$ is read as "n factorial". The reasoning used in the previous examples leads to another counting principle.

First Counting Principle for Permutations

The number of permutations of n objects is $n!$.

Note: 0! is defined to be 1, the number of ways to arrange 0 objects.

Example 2.33. The manager of a youth baseball team has chosen nine players to start a game. The total number of batting orders that is possible is the number of ways to arrange nine players in order, namely $9! = 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 362,880$. (When the authors coached youth baseball, another coach stated that he had looked at all possible batting orders and had picked the best one. Sure.) □

The previous example shows that the number of permutations of objects can be surprisingly large. Factorials grow rapidly as n increases, as shown in the following table.

n	$n!$
1	1
2	2
3	6
4	24
5	120
6	720
7	5040
8	40,320
9	362,880
10	3,628,800
11	39,916,800

The number 52! has 68 digits and is too long to bother with presenting here. This may interest card players, since 52! is the number of ways that a standard card deck can be put in order (shuffled).

Some problems involve arranging only some of the objects in order.

Example 2.34. Ten students are finalists in a scholarship competition. The top three students will receive scholarships for \$1000, \$500 and \$200. The number of ways the scholarships can be awarded is found as follows:

Rank 1	10
Rank 2	9
Rank 3	8
Total Ways to Rank	$10 \times 9 \times 8 = 720$

This is similar to Example 2.32. Any one of the 10 students can win the \$1000 scholarship. Once that is awarded, there are only 9 left for the \$500. Finally, there are only 8 left for the \$200. Note that we could also write

$$10 \times 9 \times 8 = \frac{10!}{7!} = \frac{10!}{(10-3)!}. \qquad \square$$

Example 2.34 is referred to as a problem of permuting 10 objects taken 3 at a time.

Definition 2.6. A permutation of n objects taken r at a time is an ordered arrangement of r of the original n objects, where $r \le n$.

The reasoning used in the previous example can be used to derive a counting principle for permutations.

Second Counting Principle for Permutations

The number of permutations of n objects taken r at a time is denoted by $P(n, r)$ or nPr.

$$P(n, r) = n(n-1)\cdots(n-r+1) = \frac{n!}{(n-r)!} \tag{2.9}$$

Special Cases: $P(n, n) = n!, \qquad P(n, 0) = 1$

TECHNOLOGY NOTE

Calculation of $P(n,r)$ is simple using modern calculators. Inexpensive scientific calculators typically have a factorial function key. This makes the above computation of $P(10,3)$ simple – find 10! and divide it by 7!.

More powerful calculators find quantities like $P(10,3)$ directly. For example:

(a) On the TI-30XS Multiview calculator, $P(n,r)$ can be found under PRB. If you key in 10 nPr 3, you will get the answer 720 directly.

(b) On the TI BA II Plus Professional calculator, $P(n,r)$ is available as a 2ND function on the ⊟ key.

Because modern calculators make these computations so easy, we will not avoid realistic problems in which answers involve large factorials[3].

Many computer packages will compute factorials. The spreadsheet programs that are widely used on personal computers in business also have factorial functions. For example Microsoft EXCEL has a function FACT. The corresponding function in R is FACTORIAL.

Moreover, Microsoft EXCEL has a PERMUT function to evaluate $P(n,r)$, whereas R does not directly calculate this value. $P(n,r)$ can be computed as $factorial(n)/factorial(n-r)$.

In some problems involving ordered arrangements the fact of ordering is not so obvious.

Example 2.35. The manager of a consulting firm office has 8 analysts available for job assignments. He must pick 3 analysts and assign one to a job in Bartlesville, Oklahoma, one to a job in Pensacola, Florida, and one to a job in Houston, Texas. In how many ways can he do this?

Solution. This is a permutation problem, but it is not quite so obvious that order is involved. There is no implication that the highest ranked analyst will be sent to Bartlesville. However, order is implicit in making assignment lists like this one. The manager must fill out the following form:

[3] On most calculators, factorials quickly become too large for the display mode, and factorials like 14! are given in scientific notation with some digits missing.

City	Analyst
Bartlesville	?
Pensacola	?
Houston	?

There is no implication that the order of the cities ranks them in any way, but the list must be filled out with a first choice on the first line, a second choice on the second line, and a final choice on the third line. This imposes an order on the problem. The total number of ways the job assignment can be done is

$$P(8,3) = 8 \times 7 \times 6 = \frac{8!}{5!} = 336. \qquad \square$$

2.5.6 Combinations

In every permutation problem an ordering was stated or implied. In some problems, order is not an issue.

Example 2.36.

A city council has 8 members. The council has decided to set up a committee of three members to study a zoning issue. In how many ways can the committee be selected?

Solution. This problem does not involve order, since members of a committee are not identified by order of selection. The committee consisting of Smith, Jones and London is the same as the committee consisting of London, Smith and Jones. However, there is a way to look at the problem using what we already know about ordered arrangements. If we wanted to count all the ordered selections of 3 individuals from 8 council members, the answer would be

$$P(8,3) = 336 = \text{number of ordered selections.}$$

In the 336 ordered selections, each group of 3 individuals is counted $3! = 6$ times. (Remember that 3 individuals can be ordered in $3!$ ways.) Thus the number of unordered selections of 3 individuals is

$$\frac{336}{6} = \frac{P(8,3)}{3!} = 56.$$

In the language of sets, we would say that the number of possible three-element subsets of the set of 8 council members is 56, since a subset is a selection of elements in which order is irrelevant. □

Definition 2.7. A **combination** is an unordered selection of r of the original n objects, where $r \leq n$.

The number of combinations of n objects taken r at a time is denoted by $C(n,r)$ or $\binom{n}{r}$. The notation $\binom{n}{r}$ has traditionally been more widely used, but the $C(n,r)$ notation is more commonly used in mathematical calculators and computer programs, probably because it can be typed on a single line. We will use both notations in this text.

Example 2.36 above used the reasoning that since any 3-object subset can be ordered in 3! ways, then

$$C(8,3) = \binom{8}{3} = \frac{P(8,3)}{3!}.$$

Using Equation (2.9) for $P(8,3)$, we see that $P(8,3) = \frac{8!}{5!}$ and thus

$$C(8,3) = \frac{8!}{3!5!} = \frac{8 \times 7 \times 6}{3 \times 2 \times 1} = 56.$$

This reasoning applies to the r-object subsets of any n-object set, leading to the following general counting principle:

Counting Principle for Combinations

$$\begin{aligned} \binom{n}{r} &= C(n,r) = \frac{P(n,r)}{r!} \\ &= \frac{n!}{r!(n-r)!} = \frac{n(n-1)\cdots(n-r+1)}{r!} \end{aligned} \tag{2.10}$$

Special Cases: $C(n,n) = C(n,0) = 1$

TECHNOLOGY NOTE

Any calculator with a factorial function can be used to find $C(n, r)$. The TI-30XS Multiview and TI-BA II Plus Professional calculators both have functions that calculate $C(n, r)$ directly. Microsoft EXCEL has a COMBIN function to evaluate $C(n, r)$. The equivalent in R is the *choose* function.

Example 2.37. It has become a tradition for authors of probability and statistics texts to include a discussion of their own state lottery. In the Arizona lottery, the player buys a ticket with six distinct numbers on it. The numbers are chosen from the numbers $1, 2, \ldots, 42$. What is the total number of possible combinations of 6 numbers chosen from 42 numbers?

Solution.

$$C(42, 6) = \frac{42!}{6!36!} = \frac{42 \times 41 \times 40 \times 39 \times 38 \times 37}{6!} = 5{,}245{,}786. \qquad \square$$

Example 2.38. In a new state lottery, a player will buy a ticket with six distinct numbers on it. The six winning ticket numbers are chosen from the numbers $1, 2, \ldots, N$ and $N \geq 12$. What is the maximum N for which the number of ways of getting exactly one correct number on a ticket is the same or greater than the number of ways of getting no correct numbers on the ticket?

Solution. To get exactly one correct number on a ticket means that out of six winning numbers five ticket numbers are non-winning and only one ticket number is a winning number. The total number of ways to do this is $C(N-6, 5)C(6, 1)$. Similarly, the number of ways of getting no correct numbers on the ticket is $C(N-6, 6)C(6, 0)$.

To find the maximum N for which the number of ways of getting exactly one correct number on a ticket is the same or greater than the number of ways of getting no correct numbers on the ticket, we solve the inequality by

$$C(N-6, 5)C(6, 1) \geq C(N-6, 6)C(6, 0)$$

or

$$\frac{(N-6)!}{5!(N-11)!} \times 6 \geq \frac{(N-6)!}{6!(N-12)!} \times 1.$$

Note: This only makes sense if $N \geq 12$. Then we have

$$5!(N-11)! \leq 6 \times 6!(N-12)!$$

or

$$N - 11 \leq 36$$

so that

$$N \leq 47. \qquad \square$$

2.5.7 Combined Problems

Many counting problems involve combined use of the multiplication principle, permutations, and combinations.

Example 2.39. A company has 20 male employees and 30 female employees. A grievance committee of four members is to be established. In how many ways can the committee be chosen to ensure that the majority of the members are female?

Solution. There are two ways to ensure that the majority of the committee members are females, namely a committee with three females and one male or a committee with all females. We will first use the multiplication principle to find the number of ways of obtaining the two types of committees.

We have the following two tasks to find the number of committees with three females and one male:

Task 1 – choose 3 females from 30

Task 2 – choose 1 male from 20

The number of ways to choose this committee is

$$(\text{Number of ways for Task 1}) \times (\text{Number of ways for Task 2})$$

or

$$C(30,3)C(20,1) = 4060 \times 20 = 81{,}200.$$

We have two more tasks to find the number of committees consisting of all females:

Task 3 – choose 4 females from 30

Task 4 – choose 0 male from 20

The number of ways to choose this committee is

$$(\text{Number of ways for Task 3}) \times (\text{Number of ways for Task 4})$$
$$= C(30,4)C(20,0) = 27{,}405 \times 1 = 27{,}405.$$

Then the total number of possible committees with a majority of female members is $81{,}200 + 27{,}405 = 108{,}605$. □

Example 2.40. A club has 40 members. Three of the members are running for office and will be elected president, vice-president and secretary-treasurer based on the total number of votes received. An advisory committee with 4 members will be selected from the 37 members who are not running for office. In how many ways can the club select its officers and advisory committee?

Solution. In this problem, Task 1 is to rank the three candidates for office, and Task 2 is to select a committee of 4 from 37 members. The final answer is

$$3!C(37,4) = 6 \times 66{,}045 = 396{,}270.$$ □

2.5.8 Partitions

Partitioning refers to the process of breaking a large group into separate smaller groups. The combination problems previously discussed are simple examples of partitioning problems.

Example 2.41. A company has 20 new employees to train. The company will select 6 employees to test a new computer-based training package. (The remaining 14 employees will get a classroom training course.) In how many ways can the company select the 6 employees for the new method?

Solution. The company can select 6 employees from 20 in $C(20,6) = 38,760$ ways. Each possible selection of 6 employees results in a partition of the 20 employees into two groups – 6 employees for the computer-based training and 14 for the classroom. (We would get an identical answer if we solved the problem by selection of the 14 employees for classroom training.) The number of ways to partition the group of 20 into two groups of 6 and 14 is

$$C(20,6) = C(20,14) = \frac{20!}{6!14!} = 38{,}760. \qquad \square$$

A similar pattern develops when the partitioning involves more than two groups.

Example 2.42. The company in the last example has now decided to test televised classes in addition to computer-based training. In how many ways can the group of 20 employees be divided into 3 groups with 6 chosen for computer-based training, 4 for televised classes, and 10 for traditional classes?

Solution. The partitioning requires the following two tasks:

Task 1 – select 6 of 20 for computer-based training

Task 2 – select 4 of the remaining 14 for the televised class

Once Task 2 is completed, only 10 employees will remain and they will take the traditional class. Thus the total number of ways to partition the employees is

$$C(20,6)C(14,4) = \frac{20!}{6!14!} \times \frac{14!}{4!10!} = \frac{20!}{6!4!10!} = 38{,}798{,}760. \qquad \square$$

The number of partitions of 20 objects into three groups of size 6, 4 and 10 is denoted by

$$\binom{20}{6,4,10}.$$

Example 2.42 showed that $\binom{20}{6,4,10} = \frac{20!}{6!4!10!}$. Similarly, Example 2.41 showed that $\binom{20}{6,14} = \frac{20!}{6!14!}$.

The method of Example 2.42 can be used to show that this pattern always holds for the total number of partitions.

Counting Principle for Partitions

The number of partitions of n objects into k distinct groups of sizes $n_1, n_2, \ldots, n_k$ is given by

$$\binom{n}{n_1, n_2, \ldots, n_k} = \frac{n!}{n_1!n_2!\cdots n_k!} \tag{2.11}$$

Example 2.43. An insurance company has 15 new employees. The company needs to assign 4 to underwriting, 6 to marketing, 3 to accounting, and 2 to investments. In how many different ways can this be done? (Assume that any of the 15 can be assigned to any department.)

Solution.

$$\binom{15}{4, 6, 3, 2} = \frac{15!}{4!6!3!2!} = 6{,}306{,}300.$$ □

Many counting problems can be solved using partitions if they are looked at in the right way. Exercise 2-30, finding the number of ways to rearrange the letters in the word MISSISSIPPI, is a classical problem that can be done using partitions.

2.5.9 Some Useful Identities

In Example 2.41 we noted that

$$C(20, 6) = C(20, 14) = \frac{20!}{6!14!} = 38{,}760.$$

This is a special case of the general identity $C(n, k) = C(n, n - k)$, or

$$\binom{n}{k} = \binom{n}{n-k} = \frac{n!}{k!(n-k)!}.$$

In Exercise 2-40, the reader is asked to show that the total number of subsets of an n-element set is 2^n. Since $C(n,k)$ represents the number of k-element subsets of an n-element set, we can also find the total number of subsets of an n-element set by adding up all of the $C(n,k)$.

$$2^n = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n-1} + \binom{n}{n}.$$

For example,

$$2^3 = \binom{3}{0} + \binom{3}{1} + \binom{3}{2} + \binom{3}{3} = 1 + 3 + 3 + 1.$$

In Exercise 2-39, the reader is asked to use counting principles to derive the familiar Binomial Theorem

$$(x+y)^n = \binom{n}{0} x^n + \binom{n}{1} x^{n-1}y + \binom{n}{2} x^{n-2}y^2 + \cdots + \binom{n}{n-1} xy^{n-1} + \binom{n}{n} y^n.$$

This is useful for expansions such as

$$\begin{aligned}(x+y)^4 &= \binom{4}{0} x^4 + \binom{4}{1} x^3y + \binom{4}{2} x^2y^2 + \binom{4}{3} xy^3 + \binom{4}{4} y^4 \\ &= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4.\end{aligned}$$

2.6 EXERCISES

2.2 The Language of Probability: Sets, Sample Spaces, and Events

2-1 Two dice are rolled and their sum is recorded. Let E be the event in which a sum of 8 is obtained.

(a) What is the sample space, S?

(b) List the outcomes in event E.

2-2 An insurance company provides residential wildfire coverage for homes in Arizona.

(a) What is the sample space, S, of the amount of loss?

(b) Suppose an event, E, is defined for an insurance company that offers wildfire coverage only for property values of homes in the price range of \$100,000 to \$500,000. What are the loss amounts in E?

2.3 Compound Events; Set Notation

2-3 Two dice were rolled and their sum was recorded in Exercise 2-1. List the outcomes

(a) where the two dice have identical face values;

(b) where the face value of the first die is twice the face value of the second die.

2-4 An insurance company provides residential wildfire coverage for homes in Arizona. In Exercise 2-2, event E was defined as the amount of loss for property values of homes in the price range of \$100,000 to \$500,000. Furthermore, let F be the event in which the amount of loss is greater than \$250,000.

(a) What are the elements in the event in $E \cap F$?

(b) What are the elements in the event in $E \cup F$?

(c) What are the elements in the event $\sim F$?

2.4 Set Identities

2-5 Verify the two distributive laws by drawing the appropriate Venn diagrams.

2-6 Verify De Morgan's laws by drawing the appropriate Venn diagrams.

2.5 Counting

2-7 A company has 134 employees. There are 84 who have been with the company more than 10 years, and 65 of those are college graduates. There are 23 who do not have college degrees and have been with the company less than 10 years. How many employees are college graduates?

2-8 In a survey of 185 university students, 91 were taking a history course, 75 were taking a biology course, and 37 were taking both. How many were taking a course in exactly one of these subjects?

2-9 A broker deals in stocks, bonds, and commodities. In reviewing his clients he finds that 29 own stocks, 27 own bonds, 19 own commodities, 11 own stocks and bonds, 9 own stocks and commodities, 8 own bonds and commodities, 3 own all three, and 11 have no current investments. How many clients does he have?

2-10 An insurance agent sells life, health, and auto insurance. During the year an agent met with 85 potential clients. Of these, 42 purchased life insurance, 40 health insurance, 24 auto insurance, 14 both life and health, 9 both life and auto, 11 both health and auto, and 2 purchased all three. How many of these potential clients purchased

(a) no policies;
(b) only health policies;
(c) exactly one type of insurance;
(d) life or health but not auto insurance?

2-11 A heart disease researcher has gathered data on 1400 people who have suffered heart attacks. The researcher identifies three variables associated with heart attack victims:

A – smoker, B – heavy drinker, C – sedentary life

The following data on the 1,400 victims has been gathered: 490 were smokers; 450 were heavy drinkers; 400 had a sedentary lifestyle;

- 220 were both smokers and heavy drinkers;
- 200 were both smokers and had a sedentary lifestyle;
- 200 were both heavy drinkers and had a sedentary lifestyle;
- 20 were smokers, heavy drinkers, and had a sedentary lifestyle.

Determine how many victims were

(a) neither smokers, nor heavy drinkers, nor had a sedentary lifestyle;

(b) smokers but not heavy drinkers;

(c) smokers but not heavy drinkers and did not have a sedentary lifestyle;

(d) either smokers or heavy drinkers but did not have a sedentary lifestyle.

2-12 In a survey of 120 students, the following data was obtained:

- 60 took English; 56 took Mathematics; 42 took Chemistry;
- 82 took English or Mathematics;
- 86 took English or Chemistry;
- 78 took Chemistry or Mathematics;
- 6 took all three subjects.

Find the number of students who took

(a) none of the three subjects;

(b) Mathematics only;

(c) exactly two subjects.

2-13 Suppose $n(S) = 100$, $n(A \cap B) = 25$, $n(\sim A \cap \sim B) = 16$, and $n(A) = n(B) + 31$. Find $n(A)$.

2-14 A company is surveying its workforce of 1220 people. It finds that 600 are male, 610 are college graduates, and 410 have children. Furthermore, 245 are female and college graduates, 180 are female and have children, and 240 are not graduates and have children. Lastly, 50 are female, have a college degree, and children. Find the number of employees that are female, do not have a college degree, and do not have children.

2-15 A student needs a course in each of history, mathematics, foreign languages, and economics to graduate. In looking at the class schedule he sees he can choose from 7 history classes, 8 mathematics classes, 4 foreign language classes, and 7 economics classes. In how many ways can he select the four classes he needs to graduate?

2-16 An experiment has two stages. The first stage consists of drawing a card from a standard deck. If the card is red, the second stage consists of tossing a coin. If the card is black, the second stage consists of rolling a die. How many outcomes are possible?

2-17 Let X be the n-element set $\{x_1, x_2, \ldots, x_n\}$. Show that the number of subsets of X, including X and $\emptyset$, is 2^n. (Hint: For each subset A of X, define the sequence $(a_1, a_2, \ldots, a_n)$ such that $a_i = 1$ if $x_i \in A$ and 0 otherwise. Then count the number of sequences).

2-18 An arrangement of 4 letters from the set $\{A, B, C, D, E, F\}$ is called a (four-letter) word from that set.

(a) How many four-letter words are possible if repetitions are allowed?

(b) How many four-letter words are possible if repetitions are not allowed?

2-19 Suppose any 7-digit number whose first digit is neither 0 nor 1 can be used as a telephone number. How many phone numbers are possible if repetitions are allowed? How many are possible if repetitions are not allowed?

2-20 A row contains 12 chairs. In how many ways can 7 people be seated in these chairs?

2-21 At the beginning of the basketball season a sportswriter is asked to rank the top 4 teams of the 10 teams in the XYZ conference. How many different rankings are possible?

2-22 A club with 30 members has three officers: president, secretary, and treasurer. In how many ways can these offices be filled?

2-23 The speaker's table at a banquet has 10 chairs in a row. Of the ten people to be seated at the table, 4 are left-handed and 6 are right-handed. To avoid elbowing the right-handers while eating, the left-handed people are seated in the 4 chairs on the left. In how many ways can these 10 people be seated?

2-24 Eight people are to be seated in a row of eight chairs. In how many ways can these people be seated if two of them insist on sitting next to each other?

2-25 A club with 30 members wants to have a 3-person governing board. In how many ways can this board be chosen? (Compare with Exercise 2-22.)

2-26 How many 5-card poker hands are possible from a deck of 52 cards?

2-27 How many 5-card poker hands consist of

(a) all hearts;

(b) all cards in the same suit;

(c) 2 aces, 2 kings and 1 jack?

2-28 In a class of 15 boys and 13 girls, the teacher wants a cast of 4 boys and 5 girls for a play. In how many ways can the teacher select the cast?

2-29 The Power Ball lottery uses two sets of balls: a set of white balls numbered 1 to 55 and a set of red balls numbered 1 to 42. To play, you select 5 of the white balls and 1 red ball. In how many ways can you make your selection?

2-30 How many different ways are there to arrange the letters in the word MISSISSIPPI?

2-31 An insurance company has offices in New York, Chicago, and Los Angeles. It hires 12 new actuaries and sends 5 to New York, 3 to Chicago, and 4 to Los Angeles. In how many ways can this be done?

2-32 A company has 9 analysts. It has a major project that has been divided into 3 subprojects, and it assigns 3 analysts to each task. In how ways can this be done?

2-33 Suppose that in Exercise 2-32 the company divides the 9 analysts into 3 teams of 3 each, and each team works on the whole project. In how many ways can this be done?

2-34 25 items are arranged in the following table:

A_1	A_2	A_3	A_4	A_5
A_6	A_7	A_8	A_9	A_{10}
A_{11}	A_{12}	A_{13}	A_{14}	A_{15}
A_{16}	A_{17}	A_{18}	A_{19}	A_{20}
A_{21}	A_{22}	A_{23}	A_{24}	A_{25}

Determine the number of ways to form a distinct set of three items such that no two of the items are in the same row or same column.

2-35 Suppose there are 20 males and 30 females in a club. In how many ways can you select a committee of 4 people if

(a) there are no restrictions;

(b) there must be an equal number of males and females;

(c) the majority must be female;

(d) one person is president, one is vice-president, one is secretary, and one is member-at-large;

(e) one person is president and one is vice-president?

2-36 A president, treasurer, and secretary are selected from a club with 10 members. How many arrangements are possible if

(a) there are no restrictions;

(b) Tom and Sam prefer not to serve together;

(c) Jake and Jenny must be together;

(d) Mike will only serve if he is president.

2-37 Expand $(2s - t)^4$.

2-38 In the expansion of $(2u - 3v)^8$, what is the coefficient of the term involving u^5v^3?

2-39 Prove the Binomial Theorem. (Hint: How many ways can you get the term $x^{n-k}y^k$ from the product of n factors, each of which is $(x + y)$?)

2-40 Using the Binomial Theorem, prove that the number of subsets of an n-element set is 2^n.

2.7 Sample Actuarial Examination Problems

2-41 An auto insurance company has 10,000 policyholders. Each policyholder is classified as young or old, male or female, and married or single. Of these policyholders, 3000 are young, 4600 are male, and 7000 are married. The policyholders can also be classified as 1320 young males, 3010 married males, and 1400 young married persons. Finally, 600 of the policyholders are young married males.

How many of the company's policyholders are young, female, and single?

2-42 A survey of 100 TV watchers revealed that over the last year

- 34 watched CBS,
- 15 watched NBC,
- 10 watched ABC,
- 7 watched CBS and NBC,
- 6 watched CBS and ABC,
- 5 watched NBC and ABC,
- 4 watched CBS, NBC, and ABC,
- 18 watched HGTV, and of these, none watched CBS, NBC, or ABC.

How many of the 100 TV watchers did not watch any of the four channels (CBS, NBC, ABC, or HGTV)?